Quantum correlations V

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- Quantum composite systems.

- In this lecture we describe the quantization of classical composite systems.

- We remind, classical composite systems were discussed in the second lecture.

- Analogously to description of classical composite systems we will assume only the logical independence.

- Therefore, neither dynamical nor statistical independence will not be assumed.

- Consequently, we define:

- A quantum composite system will be determined by the quadruple

$$(\mathfrak{A} \equiv \mathfrak{A}_1 \otimes \mathfrak{A}_2, \mathfrak{S} \equiv \mathfrak{S}_{\mathfrak{A}}, \{T_t\}, \varphi_0) \tag{1}$$

where $\mathfrak{A}$ (so also $\mathfrak{A}_i$) stands either for a C^*-algebra or a W^*-algebra.

- Moreover, if $\mathfrak{A}$ is a C^*-algebra (W^*-algebra) then $\mathfrak{S}$ stands for the set of all states on the global system $\mathfrak{A}$ (all density matrices - normal states on global system respectively).

- $\{T_t\}$ stands for the set of dynamical maps.

- φ_0 is a distinguished state (playing the role of distinguished probability measure).

- Frequently, it is convenient to think that the algebra $\mathfrak{A}_i$ is associated with some particular region (in $\mathbb{R}^k$), $i = 1, 2$, cf examples given in the third lecture.

- Let us consider the form of the second ingredient $\mathfrak{S}$ of the above definition (so the set of normalized, positive linear forms on $\mathfrak{A}_1 \otimes \mathfrak{A}_2$).

- The first attempt, following the classical case, would be to put $\mathfrak{S} = \mathfrak{S}_1 \otimes \mathfrak{S}_2$ or $\mathfrak{S} = \overline{conv}(\mathfrak{S}_1 \otimes \mathfrak{S}_2)$.

- Surprisingly these sets do not contain all states.

- Namely, one has (see exercise 11.5.11 Kadison-Ringrose book!)

- **Example 1.** *Let* $\mathfrak{A}_1 = B(\mathcal{H})$ *and* $\mathfrak{A}_2 = B(\mathcal{K})$ *where* $\mathcal{H}$ *and* $\mathcal{K}$ *are 2-dimensional Hilbert spaces. Consider the vector state* $\omega_x(\cdot) = (x, \cdot\, x)$ *with* $x = \frac{1}{\sqrt{2}}(e_1 \otimes f_1 + e_2 \otimes f_2)$ *where* $\{e_1, e_2\}$ *and* $\{f_1, f_2\}$ *are orthonormal bases in* $\mathcal{H}$ *and* $\mathcal{K}$ *respectively. Let* ρ *be any state in the norm closure of the convex hull of product states, i.e.* $\rho \in \overline{conv}(\mathfrak{S}_1 \otimes \mathfrak{S}_2)$*. Then, one can show that*

$$\|\omega_x - \rho\| \geq \frac{1}{4}. \tag{2}$$

- **Remark 2.** *One should note that* ω_x *can always be approximated by a finite linear combination of simple tensors (as it was explained in the fourth lecture). However, here we wish to approximate* ω_x *by a convex combination of positive (normalized) functionals and this makes the difference.*

- Consequently, contrary to the classical case even in the simplest non-commutative case, the space of all states of $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ is not norm closure of $conv(\mathfrak{S}_1 \otimes \mathfrak{S}_2)$.

- *It means, in mathematical terms, that for non-commutative case the weak* Riemann approximation property of a (classical) measure does not hold, in general!*

- Thus, it is natural to distinguish states having analogous form to that appearing for classical composite systems.

- Hence we have:

- **Definition 3.** – *C^*-algebra case.*
  Let $\mathfrak{A}_i$, $i = 1, 2$ be a C^*-algebra, $\mathfrak{S}$ the set of all states on $\mathfrak{A} \equiv \mathfrak{A}_1 \otimes \mathfrak{A}_1$, i.e. the set of all normalized positive forms on $\mathfrak{A}$. The subset $\overline{conv}(\mathfrak{S}_1 \otimes \mathfrak{S}_2)$ in $\mathfrak{S}$ will be called the set of separable states and will be denoted by $\mathfrak{S}_{sep}$. The closure is taken with respect to the norm of $\mathfrak{A}^*$. The subset $\mathfrak{S} \setminus \mathfrak{S}_{sep} \subset \mathfrak{S}$ is called the subset of entangled states.
  – *W^*-algebra case.*
  Let $\mathfrak{M}_i$, $i = 1, 2$ be a W^*-algebra, $\mathfrak{M} = \mathfrak{M}_1 \otimes \mathfrak{M}_2$ be the spacial tensor product of $\mathfrak{M}_1$ and $\mathfrak{M}_2$, $\mathfrak{S}$ the set of all states on $\mathfrak{M}$, and $\mathfrak{S}^n$ the set of all normal states on $\mathfrak{M}$, i.e. the set of all normalized, weakly-continuous positive forms on $\mathfrak{M}$ (equivalently, the set of all density matrices). The subset $\overline{conv}^{\pi}(\mathfrak{S}_1^n \otimes \mathfrak{S}_2^n)$ in $\mathfrak{S}^n$ will be called the set of separable states and will be denoted by $\mathfrak{S}_{sep}^n$. The closure is taken with respect to the operator space projective norm on $\mathfrak{M}_{1,*} \odot \mathfrak{M}_{2,*}$. The subset $\mathfrak{S}^{\mathfrak{n}} \setminus \mathfrak{S}_{sep}^n \subset \mathfrak{S}^{\mathfrak{n}}$ is called the subset of normal entangled states.*

- **Remark 4.** *1. As a separable state has the form of an arbitrary classical state (presented in the second lecture), it is naturally to adopt the convention that* $\mathfrak{S}_{sep}$ ($\mathfrak{S}^n_{sep}$) *contains only classical correlations.*

  2. The set of entangled states is the set where the quantum (so extra) correlations can occur.

  3. The difference between C^**-algebra case and* W^**-algebra case stems from the Grothendieck's theory of tensor products.*

  4. In particular, the exposition given in the last lecture shows how naturally the projective tensor product is appearing in Definition 3.

5. *Furthermore, to appreciate the use of projective topology, we recall that the set of all normal states on* $\mathfrak{M}$ *is weakly-*$*$ *dense in the set of all states on* $\mathfrak{M}$*; see Bratteli, Robinson book.*

6. *The principal significance of normality of a state, from physical point of view, follows from existence of the number operator, see vol II of Bratteli, Robinson book. In other words, a normal state corresponds to such situation when the number of particles has well defined sense.*

- Having defined separable, and entangled states we turn to the question why other arguments, leading to the general form of two points correlation function, are not working in the non-commutative setting.

- Our first observation is the following:

- **Fact 5.** *1. classical case.*

Let δ_a be a Dirac's measure on a product measure space, i.e. δ_a is given on $\Gamma_1 \times \Gamma_2$. Note that the marginal of the point measure δ_a gives another point measure, i.e. $\delta_a|_{\Gamma_1} = \delta_{a_1}$. Here we put $a \in \Gamma_1 \times \Gamma_2$, $a = (a_1, a_2)$. The same in "physical terms" reads: a reduction of a pure state is again a pure state.

2. *non-commutative case.*

Let $\mathcal{H}$ and $\mathcal{K}$ are finite dimensional Hilbert spaces. Without loss of generality we can assume that dim$\mathcal{H}$=dim$\mathcal{K} = n$.

Let $\omega_x(\cdot) = (x, \cdot\, x)$ be a state on $B(\mathcal{H}) \otimes B(\mathcal{K})$ where x is assumed to be of the form

$$x = \frac{1}{\sqrt{n}} \left(\sum_i e_i \otimes f_i \right) \tag{3}$$

(so $\omega_x(\cdot)$ is a fully entangled state).

Here $\{e_i\}$ and $\{f_i\}$ are basis in $\mathcal{H}$ and $\mathcal{K}$ respectively.

Then, we have

$$\omega_x\left(A\otimes\mathbb{1}\right)=\frac{1}{n}\left(\sum_i e_i\otimes f_i, A\otimes\mathbb{1}\sum_j e_j\otimes f_j\right)$$
$$=\frac{1}{n}\sum_{i,j}\left(e_i, Ae_j\right)\left(f_i, f_j\right)=\mathrm{Tr}_{\mathcal{H}}\frac{1}{n}\mathbb{1}A\equiv\mathrm{Tr}_{\mathcal{H}}\varrho_0 A, \quad (4)$$

where $\varrho_0=\frac{1}{n}\mathbb{1}$ is "very non pure" state.

3. *In other words, the non-commutative counterpart of the marginal of a point measure (pure state) does not need to be again a point measure (pure state).*
4. *Consequently, the crucial ingredient of the discussion, given in the second lecture, leading to the general form of two point (classical) correlation function is not valid in non-commutative case.*

- The next difficulty follows from the geometrical characterization of the set of states.
- Namely, in geometrical description of a convex set in finite dimensional spaces one can distinguish two kinds of convex closed sets: simplexes and non-simplexes. Relevant illustrations were given in he first lectures!.
- Let K be a convex compact set. From Krein-Milman theorem, the set K has extreme points $\{k_i\}$ and $K = \overline{conv}\,\{k_i\}$.
- Thus K is a convex hull of its extreme points $\{k_i\}$.
- If any point of K can be given uniquely as convex combination of extreme points then K is called a simplex.
- For example, a triangle is a simplex, but a circle is not a simplex. This gives an intuition.

- However, here, considering infinite dimensional algebras, so genuine quantum systems!, and being interested in certain subsets of positive forms on these algebras, the definition of Choquet is more suitable.

- *Let K be a base of a convex cone C with apex at the origin. The cone C gives rise to the order $\leq$ ($a \leq b$ if and only if $b - a \in C$). K is said to be a simplex if C equipped with the order $\leq$ is a lattice.*

- (Lattice is a partially ordered set in which every two elements have a supremum and an infimum).

- The importance of this notion follows from the well known result saying that in the classical case, the set of all states forms a simplex while this is not true for the quantum case. More precisely:

- **Proposition 6.** *Let $\mathfrak{A}$ be a C*-algebra. Then the following conditions are equivalent*

  1. *The state space $\mathfrak{S}_{\mathfrak{A}}$ is a simplex.*
  2. *$\mathfrak{A}$ is abelian algebra.*
  3. *Positive elements $\mathfrak{A}^+$ of $\mathfrak{A}$ form a lattice.*

- Therefore in quantum case the set of states is not a simplex (contrary to the classical case).

- Consequently, in quantum case, all possible decompositions of a given state should be taken into account.

- This should make clear why decomposition theory is absolutely necessary.

- We wish to close this section with a basic package of terminology used in open systems and quantum information theory (so also in an analysis of quantum composite systems).

- On the one hand, this is a feature of quantum composite systems. On the other hand, we will need material for an analysis of PPT states.

- A linear map $\alpha : B(\mathcal{H}) \to B(\mathcal{K})$ is called k-positive if a map $id_{M_k} \otimes \alpha : M_k \otimes B(\mathcal{H}) \to M_k \otimes B(\mathcal{K})$ is positive, where $M_k \equiv M_k(\mathbb{C})$ denotes the algebra of $k \times k$ matrices with complex entries.

- A map α is called completely positive if it is k-positive for any k.

- A completely positive map α will be shortly called a CP map.

- A positive map $\alpha : B(\mathcal{H}) \longrightarrow B(\mathcal{K})$ is called *decomposable* if there are completely positive maps $\alpha_1, \alpha_2 : B(\mathcal{H}) \longrightarrow B(\mathcal{K})$ such that $\alpha = \alpha_1 + \alpha_2 \circ \tau_{\mathcal{H}}$, where $\tau_{\mathcal{H}}$ stands for a transposition map on $B(\mathcal{H})$.

- Let $\mathcal{P}$, $\mathcal{P}_c$ and $\mathcal{P}_d$ denote the set of all positive, completely positive and decomposable maps from $B(\mathcal{H})$ to $B(\mathcal{K})$, respectively.

- Note that

$$\mathcal{P}_c \subset \mathcal{P}_d \subset \mathcal{P}$$

- Finally, let us define the family of PPT (transposable) states on $B(\mathcal{H}) \otimes B(\mathcal{K})$

$$\mathfrak{S}_{PPT} = \{\varphi \in \mathfrak{S} : \varphi \circ (id_{B(\mathcal{H})} \otimes \tau_{\mathcal{K}}) \in \mathfrak{S}\}. \tag{5}$$

where, as before, $\tau_{\mathcal{K}}$ stands for the transposition map, now defined on $B(\mathcal{K})$.

- $\tau_{\mathcal{K}}$ - the transposition map - is a positive map but not completely positive.

- Consequently, the condition appearing in the definition of PPT states is not trivial!.

- Note that due to the positivity of the transposition $\tau_{\mathcal{K}}$ every separable state φ is transposable, so

$$\mathfrak{S}_{sep} \subset \mathfrak{S}_{PPT} \subset \mathfrak{S}.$$